ON MODULI DESCRIPTION OF LOCAL MODELS FOR RAMIFIED UNITARY GROUPS AND RESOLUTION OF SINGULARITY

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Abstract

Local models are schemes which are intended to model the étale-local structure of p-adic integral models of Shimura varieties. In the setting of local models for ramified unitary groups, Smithling has proposed a further refinement to the moduli problems in Pappas and Rapoport's work to characterize the local model. In this paper we examine a special case with signature $(n-1, 1)$ under this construction, and propose a moduli description for resolution of singularities.

READERS: Professor Brian D. Smithling (Advisor), Professor David Savitt

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Moduli Description and Structure of the Local Model

1.1 Introduction

For the study of Shimura varieties, it is of interest to have a model over the ring of integers $\mathcal{O}_E$, where E is the completion of the reflex field at a finite prime of residue characteristic p. To qualify as a model, it should be flat and have only mild singularities. In the case of PEL-type Shimura varieties, such models can be defined by posing the moduli problem over $\mathcal{O}_E$, and if the parahoric level structure at p is defined as the stabilizer of a self-dual lattice chain, the questions of local nature can be reduced to the corresponding local models. They coincide with the integral models locally for the étale topology, but can be studied more easily.

Such models and local models have been given by Rapoport and Zink in terms of explicit moduli problems, from the abelian varieties describing the Shimura varieties. In [1, 2], Görtz showed that the local model is flat when the group describing the Shimura variety splits over an unramified extension of $\mathbb{Q}_p$ and only involves types A and C. However, as observed by Pappas [5], when the group is a ramified, quasi-split unitary group, the local model is not always flat. In [4], Pappas and Zhu gave the honest local models by a group-theoretic construction of the flat closure of the generic fiber in the Rapoport-Zink local model (the naive local model).

When the naive local model is not already flat, it remains important to construct a moduli description of the local model, since the flat closure construction does not give one. The case of ramified, quasi-split unitary group is in our particular interest and a lot of results have been achieved.

Let F_0 be a complete discretely valued field with perfect residue field of characteristic not 2, and F/F_0 be a ramified quadratic extension. Let $n \geq 2$ be an integer, and $m := \lfloor n/2 \rfloor$. Let $r + s = n$ be a partition of n and call (r, s) the signature. Let $I \subset \{0, \cdots, m\}$ be a nonempty subset with the property that

$$n \text{ is even and } m - 1 \in I \Longrightarrow m \in I. \tag{1.1.1}$$

Let M_I^{naive} denote the naive local model attached to these data. Pappas introduced the wedge condition in [5], defining the wedge local model $M_I^{\wedge}$ as the closed subscheme cut out in the naive local model by the wedge condition, and conjectured that it coincides with the local model in the $I = \{0\}$ case. For other cases Pappas and Rapoport later introduced the spin condition in [6], cutting out a closed subscheme M_I^{spin} inside the wedge local model. They conjectured that the spin local models defined by the wedge and spin conditions are always flat and proved some low-dimensional cases. Smithling is proposing a proof of the conjecture of Pappas in [12], and by showing the topological flatness of the spin local models in [9] and [10], he proved the conjecture of Pappas and Rapoport on the level of topological spaces.

However, in [11] Smithling constructed a counterexample in which the spin local model is not flat for some odd n, so in general the conjecture does not hold. In response to this counterexample he proposed a further refinement, the strengthened spin condition, which at least fixes the flatness in this example. He then conjectured that the strengthened spin local model M_I is always flat.

Conjecture 1 (Smithling [11])**.** *For any signature and nonempty I satisfying* (1.1.1)*, M_I is flat over* Spec $\mathcal{O}_E$.

For even $n = 2m$ and signature $(n-1, 1)$, Rapoport, Smithling and Zhang showed in [8] that if $m - 1 \in I$ and $m \notin I$, $M_{I \cup \{m\}}$ and M_I are isomorphic, so for $I = \{m-1, m\}$, it suffices to study $M_{\{m-1\}}$ and we prove Conjecture 1 in this case.

Theorem 2. *For even n and signature $(n-1, 1)$, $M_{\{m-1,m\}} = M^{loc}_{\{m-1,m\}}$. The special fiber is reduced with 3 normal irreducible components. $M_{\{m-1,m\}}$ is regular outside one point.*

In §2 we recall the definition of the naive, wedge, spin and strengthened spin local models. In §3 we specialize to the case of $n = 2m \geq 4, I = \{m-1, m\}, (r, s) = (n-1, 1)$, and reduce the proof of the flatness in Theorem 2 to Proposition 1. In §4 we specifically examine the spin conditions on the special fiber and thus prove Proposition 1. In §5 we prove the rest of Theorem 2.

1.2 The moduli problem

In this section we recall the moduli construction of the naive local model M_I^{naive} for ramified unitary groups, and the wedge and spin conditions cutting out $M_I^{\wedge}$, M_I^{spin}, and M_I.

1.2.1 Standard lattices

Throughout this paper F_0 is a discretely valued non-Archimedean field with ring of integers $\mathcal{O}_{F_0}$, uniformizer π_0 and residue field k of characteristic $\neq 2$. F is a ramified quadratic extension of F_0 with ring of integers $\mathcal{O}_F$, uniformizer π satisfying $\pi^2 = \pi_0$, and the same residue field k.

Let n be an integer ≥ 2, and $m = \lfloor \frac{n}{2} \rfloor$. For any $1 \leq i \leq n$, let $i^\vee := n + 1 - i$, and for any $1 \leq i \leq 2n$, let $i^* := 2n + 1 - i$.

In the vector space F^n, consider

$$\phi : F^n \times F^n \to F,$$

the $(F/F_0)-$Hermitian form which we may assume to be split with respect to the standard basis $e_1, \ldots, e_n$, i.e.

$$\phi(ae_i, be_j) = \bar{a}b\delta_{ij^\vee}, \text{ for all } i, j = 1, \ldots, n, \text{ and } a, b \in F$$

where $a \mapsto \bar{a}$ is the nontrivial element of $\mathrm{Gal}(F/F_0)$.

ϕ induces two F_0-bilinear forms

$$\langle x, y \rangle := \frac{1}{2}\mathrm{Tr}_{F/F_0}(\pi^{-1}\phi(x, y)) \text{ and } (x, y) := \frac{1}{2}\mathrm{Tr}_{F/F_0}(\phi(x, y))$$

The form $\langle\, ,\, \rangle$ is alternating, and $(\, ,\,)$ is symmetric.

For any $\mathcal{O}_F$-lattice Λ in F^n, let

$$\hat{\Lambda} := \{x \in F^n | \phi(x, \Lambda) \subset \mathcal{O}_F\} = \{x \in F^n | \langle x, \Lambda \rangle \subset \mathcal{O}_{F_0}\}$$

be the $\langle\, ,\, \rangle$-dual lattice, and

$$\hat{\Lambda}^s := \{x \in F^n | (x, \Lambda) \subset \mathcal{O}_{F_0}\}$$

be the $(\, ,\,)$-dual of Λ. We have $\hat{\Lambda}^s = \pi^{-1}\hat{\Lambda}$.

In particular, from now on we work with the standard $\mathcal{O}_F$-lattices

$$\Lambda_i := \Sigma_{l=1}^{j}\pi^{-a-1}\mathcal{O}_F e_l + \Sigma_{l=j+1}^{n}\pi^{-a}\mathcal{O}_F e_l \subset F^n$$

for each integer $i = na + j$ with $0 \le j < n$.

Then for all i, $\hat{\Lambda}_i = \Lambda_{-i}$, $\hat{\Lambda}_i^s = \Lambda_{n-i}$, and $\langle\ ,\ \rangle$ defines a perfect $\mathcal{O}_{F_0}$-bilinear pairing

$$\Lambda_i \times \Lambda_{-i} \longrightarrow \mathcal{O}_{F_0}$$

and $(\ ,\ )$ defines a perfect $\mathcal{O}_{F_0}$-bilinear pairing

$$\Lambda_i \times \Lambda_{n-i} \longrightarrow \mathcal{O}_{F_0}.$$

With respect to both $(\ ,\ )$ and $\langle\ ,\ \rangle$, the Λ_i's form a complete, periodic, self-dual lattice chain

$$\cdots \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots .$$

1.2.2 Naive local model and local model

Let $I \subset \{0, \cdots, m\}$ be a nonempty set satisfying (1.1.1), and $r + s = n$ is a partition of n. The reflex field E is F if $r \neq s$ or F_0 if $r = s$.

The naive local model M_I^{naive} is the moduli problem on the category of $\mathcal{O}_E$-algebras which associates each $\mathcal{O}_E$-algebra R with the families

$$(\mathcal{F}_i \subset \Lambda_i \otimes_{\mathcal{O}_{F_0}} R)_{i \in \pm I + n\mathbb{Z}}$$

such that for each such i,

(LM1) $\mathcal{F}_i$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-submodule of $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R$ and a direct summand of rank n;

(LM2) For each $i < j$, the morphism $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R \to \Lambda_j \otimes_{\mathcal{O}_{F_0}} R$ maps $\mathcal{F}_i$ into $\mathcal{F}_j$:

$$\begin{array}{ccc} \Lambda_i \otimes_{\mathcal{O}_{F_0}} R & \longrightarrow & \Lambda_j \otimes_{\mathcal{O}_{F_0}} R \\ \cup & & \cup \\ \mathcal{F}_i & \longrightarrow & \mathcal{F}_j \end{array}$$

(LM3) The isomorphism $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R \longrightarrow \Lambda_j \otimes_{\mathcal{O}_{F_0}} R$ induced by $\Lambda_i \xrightarrow{\pi \otimes 1} \Lambda_{i-n}$ identifies $\mathcal{F}_i$ with $\mathcal{F}_{i-n}$;

(LM4) The R-bilinear perfect pairing

$$(\Lambda_i \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_{n-i} \otimes_{\mathcal{O}_{F_0}} R) \longrightarrow R$$

induced by $\Lambda_i \times \Lambda_{n-i} \xrightarrow{(\ ,\ )} \mathcal{O}_{F_0}$ identifies $\mathcal{F}_{n-i}$ with $\mathcal{F}_i^{\perp}$;

(LM5) (Kottwitz condition) Multiplication with $\pi \otimes 1$ acts on $\mathcal{F}_i$ as an R-linear endomorphism with characteristic polynomial

$$\det\ (T \cdot id - \pi \otimes 1 | \mathcal{F}_i) = (T + \pi)^r (T - \pi)^s \in R[T].$$

when $r = s$, the polynomial still makes sense since it can be interpreted as $(T^2 - \pi_0)^r \in R[T]$ where R is any $\mathcal{O}_{F_0}$-algebra.

It is clear that the moduli problem is represented by a projective $\mathcal{O}_E$-scheme, since (LM1)-(LM5) define a closed subfunctor of a product of finitely many $\mathrm{Gr}(n, \Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E), 0 \leq i < n$. One can also show the generic fiber is simply $\mathrm{Gr}_{\mathcal{O}_E}(r, n)$.

The local model M_I^{loc} is defined as the flat scheme-theoretic closure in M_I^{naive} of the generic fiber.

1.2.3 The wedge condition and spin condition

The wedge condition on an R-point $(\mathcal{F}_i)_i \in M_I^{\mathrm{naive}}(R)$ is for each i,

(LM6) if $r \neq s$,

$$\bigwedge_R^{r+1} (\pi \otimes 1 - 1 \otimes \pi | \mathcal{F}_i) = 0, \text{ and } \bigwedge_R^{s+1} (\pi \otimes 1 + 1 \otimes \pi | \mathcal{F}_i) = 0.$$

The wedge local model $M_I^{\wedge}$ is the closed subscheme of M_I^{naive} satisfying (LM6).

To state the spin conditions, if $r \neq s$ we consider the F-vector space

$$V := F^n \otimes_{F_0} F$$

with F acting on the right factor. Now $\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$ is an $\mathcal{O}_F$-lattice in V. Let

$$W := \bigwedge_F^n V.$$

By the definition of ϕ, $(\ ,\ )$ is either already split over F^n (if n is even) or becomes split after tensoring with F(if n is odd). In all cases there is an F-basis $f_1, \ldots, f_{2n}$ such that $(f_i, f_j) = \delta_{ij^*}$. Hence the $SO((\ ,\ ))(F) \simeq SO_{2n}(F)$-representation on W has a decomposition

$$W = W_1 \oplus W_{-1}.$$

In fact, if $f_1, \dots, f_{2n}$ is a split F-basis of V, for $S = \{i_1 < \dots i_n\} \subset \{1, \dots, 2n\}$, let

$$f_S := f_{i_1} \wedge \dots \wedge f_{i_n} \in W.$$

The f_S's form a basis of W, and let σ_S be the permutation on $\{1, \dots, 2n\}$ sending $\{1. \cdots, n\}$ to S in increasing order and $\{n+1, \dots, 2n\}$ to $\{1, \dots, 2n\} \setminus S$ in increasing order. For any such S, write $S^* = \{i^* | i \in S\}$ and $S^\perp = \{1, \dots, 2n\} \setminus S^*$. We define an F-linear operator on W by

$$a(f_S) := \mathrm{sgn}(\sigma_S) f_{S^\perp}.$$

Then $W_{\pm 1}$ is the ± 1-eigenspace for a:

$$W_{\pm 1} = \mathrm{span}_F \{f_S \pm \mathrm{sgn}(\sigma_S) f_{S^\perp} | \#S = n\}.$$

One sees that $W_{\pm 1}$ is not independent of choices of the split basis, for a determinant -1 orthogonal transformation sends a split basis to another split basis but interchanges W_{-1} and W_1. But any two split bases at most differ by an orthogonal transformation, hence W_{-1} and W_1 are well-defined up to labeling. We fix one split basis $f_1, \dots, f_{2n}$ to define the spin conditions.

If $n = 2m$ is even, we take

$$-\pi^{-1} e_1 \otimes 1, \dots, -\pi^{-1} e_m \otimes 1, e_{m+1} \otimes 1, \dots, e_n \otimes 1, e_1 \otimes 1, \dots, e_m \otimes 1, \pi e_{m+1} \otimes 1, \dots, \pi e_n \otimes 1. \quad (1.2.1)$$

If $n = 2m + 1$ is odd, we take

$$\begin{aligned} -\pi^{-1} e_1 \otimes 1, \dots, -\pi^{-1} e_m \otimes 1, e_{m+1} \otimes 1 - \pi e_{m+1} \otimes \pi^{-1}, e_{m+2} \otimes 1, \dots, e_n \otimes 1, \\ e_1 \otimes 1, \dots, e_m \otimes 1, \frac{e_{m+1} \otimes 1 + \pi e_{m+1} \otimes \pi^{-1}}{2}, \pi e_{m+2} \otimes 1, \dots, \pi e_n \otimes 1. \end{aligned} \quad (1.2.2)$$

For the $\mathcal{O}_F$-lattice, $\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F \subset V$, let

$$W(\Lambda_i) := \bigwedge_{\mathcal{O}_F}^{n} (\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$$

be the correspondent $\mathcal{O}_F$-lattice in W, and

$$W(\Lambda_i)_{\pm 1} := W_{\pm 1} \cap W(\Lambda_i).$$

The spin condition on an R-point $(\mathcal{F}_i)_i \in M_I^{\mathrm{naive}}(R)$ is for each i,

if $r \neq s$,

$$\bigwedge_R^n \mathcal{F}_i \subset \mathrm{im}[W(\Lambda_i)_{(-1)^s} \otimes_{\mathcal{O}_F} R \longrightarrow W(\Lambda_i) \otimes_{\mathcal{O}_F} R]; \tag{1.2.3}$$

if $r = s$, $(\ ,\ )$ is already split over F_0, the spin condition can be formulated directly over F_0: take $W = \bigwedge_{F_0}^n F^n$, $W(\Lambda_i) = \bigwedge_{\mathcal{O}_{F_0}}^n \Lambda_i$, then

$$\bigwedge_R^n \mathcal{F}_i \subset \mathrm{im}[W(\Lambda_i)_{(-1)^s} \otimes_{\mathcal{O}_{F_0}} R \longrightarrow W(\Lambda_i) \otimes_{\mathcal{O}_{F_0}} R]$$

The spin local model M_I^{spin} is the closed subscheme of $M_I^{\wedge}$ satisfying (1.2.3).

1.2.4 The strengthened spin condition

We finally turn to the strengthened spin condition. Considered in the F-vector space V, $\pi \otimes 1$ acts semisimply with eigenvalues π and $-\pi$ each n times. Let V_π and $V_{-\pi}$ denote the respective eigenspaces. Then we can also decompose W into

$$W = \bigoplus W^{r,s},$$

where

$$W^{r,s} := \bigwedge_F^r V_{-\pi} \wedge_F \bigwedge_F^s V_\pi.$$

Let

$$W_{\pm 1}^{r,s} := W^{r,s} \cap W_{\pm 1}.$$

For the $\mathcal{O}_F$-lattice $\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F \subset V$, let

$$W(\Lambda)_{\pm 1}^{r,s} := W_{\pm 1}^{r,s} \cap W(\Lambda).$$

The strengthened spin condition on an R-point $(\mathcal{F}_i)_i \in M_I^{\mathrm{naive}}(R)$ is for each i,

(LM7) if $r \neq s$,

$$\bigwedge_R^n \mathcal{F}_i \subset \mathrm{im}[W(\Lambda_i)_{(-1)^s}^{r,s} \otimes_{\mathcal{O}_F} R \longrightarrow W(\Lambda_i) \otimes_{\mathcal{O}_F} R];$$

if $r=s$, the condition is again defined over $\mathcal{O}_{F_0}$: take $W=\bigwedge_{F_0}^n F^n$, $W(\Lambda_i)=\bigwedge_{\mathcal{O}_{F_0}}^n \Lambda_i$, then

$$\bigwedge_R^n \mathcal{F}_i \subset \operatorname{im}[W(\Lambda_i)^{r,s}_{(-1)^s}\otimes_{\mathcal{O}_{F_0}} R \longrightarrow W(\Lambda_i)\otimes_{\mathcal{O}_{F_0}} R]$$

The strengthened spin local model M_I is the closed subscheme of $M_I^{\wedge}$ satisfying (LM7).

If R is an F-algebra, clearly (LM1)-(LM5) automatically imply (LM6)-(LM7), so all the closed conditions are naturally satisfied on the generic fiber. We obtain the chain of closed immersions

$$M_I^{\text{loc}} \subset M_I \subset M_I^{\text{spin}} \subset M_I^{\wedge} \subset M_I^{\text{naive}}$$

where all the generic fibers are equal.

1.3 Flatness of $M_{\{m-1,m\}}$ for even n and signature $(n-1,1)$

We prove Theorem 2 in this and the next section. For the rest of the paper $n=2m\geq 4$ and $(r,s)=(n-1,1)$.

First, as a special case of the result in [8], the forgetful functor $M_{\{m-1,m\}}\to M_{\{m-1\}}$ is an isomorphism for signature $(n-1,1)$. Therefore it is equivalent to check whether the strengthened spin condition provides the moduli description for the local model $M^{\text{loc}}_{\{m-1\}}$, although $I=\{m-1\}$ does not satisfy (1.1.1).

We want to show the first closed embedding in $M^{\text{loc}}_{\{m-1\}}\subset M_{\{m-1\}}\subset M^{\text{spin}}_{\{m-1\}}$ is an equality. Their generic fibers are the same, hence it comes down to show the equality of special fibers. Furthermore, Smithling has proved the topological flatness of spin local models in [9, 10], in other words all three are equal as topological spaces. So if $M_{\{m-1\}}$ has reduced special fiber, the equality will be forced to hold. In [6] Pappas and Rapoport construct a closed embedding of the geometric special fiber of the naive local model into an affine flag variety associated to GU_n, where the image is a union of Schubert varieties, and the geometric special fiber of the local model contains the Schubert varieties over elements in the μ-admissible set. In our case, the geometric special fiber of $M^{\text{loc}}_{\{m-1\}}$ and $M_{\{m-1\}}$ topologically contain the same Schubert cells, including the unique closed Schubert cell, which is the image of the "worst point"

$$((\pi\otimes 1)\cdot(\Lambda_{m-1}\otimes_{\mathcal{O}_{F_0}} k)\subset \Lambda_{m-1}\otimes_{\mathcal{O}_{F_0}} k, (\pi\otimes 1)\cdot(\Lambda_{m+1}\otimes_{\mathcal{O}_{F_0}} k)\subset \Lambda_{m+1}\otimes_{\mathcal{O}_{F_0}} k).$$

An open neighborhood of this point thus intersects with every other Schubert cell, then it remains to show that the special fiber is also reduced in a neighborhood of the worst point.

$M_{\{m-1\}}$ is a closed subscheme of $\mathrm{Gr}(n, \Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$, since $\mathcal{F}_{m-1}$ determines all the other $\mathcal{F}_i$ by periodic and perpendicular conditions. On the special fiber, the worst point sits inside the standard affine open in $\mathrm{Gr}(n, \Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} k)$ represented by $2n \times n$ matrices

$$\begin{pmatrix} X \\ I_n \end{pmatrix}$$

with respect to the standard basis of $\Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} k$

$$\begin{aligned} \pi^{-1}e_1 \otimes 1, \dots, \pi^{-1}e_{m-1} \otimes 1, e_m \otimes 1, \dots, e_n \otimes 1, \\ e_1 \otimes 1, \dots, e_{m-1} \otimes 1, \pi e_m \otimes 1, \dots, \pi e_n \otimes 1, \end{aligned} \tag{1.3.1}$$

where the worst point corresponds to $X = 0$.

In this affine chart (LM1)-(LM7) translate to some matrix identities of X, with which we hope to obtain the local chart around the worst point, and thus prove flatness part of Theorem 2 by establishing

Proposition 1. *The special fiber of $M_{\{m-1\}}$ is reduced.*

1.4 The special fiber of $M_{\{m-1\}}$

In this section we compute the affine chart on the special fiber of M_{m-1} around the worst point, the k-point

$$((\pi \otimes 1) \cdot (\Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} k) \subset \Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} k, (\pi \otimes 1) \cdot (\Lambda_{m+1} \otimes_{\mathcal{O}_{F_0}} k) \subset \Lambda_{m+1} \otimes_{\mathcal{O}_{F_0}} k)$$

which becomes the unique closed Schubert cell when one embeds the geometric special fiber into an affine flag variety. As discussed in §, this means to find the k-scheme of $2n \times n$ matrices $\begin{pmatrix} X \\ I_n \end{pmatrix}$ satisfying (LM1)-(LM7). We know (LM7) implies (LM5), so only (LM1)-(LM4), (LM6) and (LM7) need to be applied.

Throughout this section R is a k-algebra, and to lighten calculation, we take the $\mathcal{O}_{F_0}$-bases

$$e_{m+2}, \dots, e_n, \pi^{-1}e_1, \dots, \pi^{-1}e_{m-1}, e_m, e_{m+1}, \pi e_{m+2}, \dots, \pi e_n, e_1, \dots, e_{m-1}, \pi e_m, \pi e_{m+1} \tag{1.4.1}$$

$$e_{m+2},\dots,e_n,\pi^{-1}e_1,\dots,\pi^{-1}e_{m-1},\pi^{-1}e_m,\pi^{-1}e_{m+1},\pi e_{m+2},\dots,\pi e_n,e_1,\dots,e_{m-1},e_m,e_{m+1} \tag{1.4.2}$$

for Λ_{m-1} and Λ_{m+1} respectively, and in what follows write $\begin{pmatrix} X \\ I_n \end{pmatrix}$ as the matrices with respect to (1.4.1) instead of (1.3.1).

1.4.1 Basic conditions and wedge condition

Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, X_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where X_1 is of size $(2m-2)\times(2m-2)$, X_2 is of size $(2m-2)\times 2$, X_3 is of size $2\times(2m-2)$, X_4 is of size 2×2, and A,B,C,D are of size $(m-1)\times(m-1)$.

On the special fiber, the map $\Lambda_{m-1}\otimes_{\mathcal{O}_{F_0}} R \to \Lambda_{m+1}\otimes_{\mathcal{O}_{F_0}} R$ and $\Lambda_{m+1}\otimes_{\mathcal{O}_{F_0}} R \to \Lambda_{n+m-1}\otimes_{\mathcal{O}_{F_0}} R$ are represented by the matrices

$$A_{m-1} = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n-2} & 0 \\ 0 & I_2 & 0 & 0 \end{pmatrix} \text{ and } A_{m+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ I_{n-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix}$$

respectively.

The pairing $(\ ,\ )\otimes_{\mathcal{O}_{F_0}} R : (\Lambda_{m+1}\otimes_{\mathcal{O}_{F_0}} R)\times(\Lambda_{m-1}\otimes_{\mathcal{O}_{F_0}} R) \longrightarrow R$ is represented by the matrix

$$M = \begin{pmatrix} 0 & 0 & J_{n-2} & 0 \\ 0 & 0 & 0 & -H_2 \\ -J_{n-2} & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \end{pmatrix}$$

where H_l is the $l\times l$-matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ and $J_{2l} = \begin{pmatrix} & H_l \\ -H_l & \end{pmatrix}$.

$\pi \otimes 1$-stability on $\mathcal{F}_{m-1}$ translates to

$$(\pi \otimes 1) \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} T$$

for some T, so $T = X, X^2 = 0$, i.e.

$$\begin{pmatrix} X_1^2 + X_2X_3 & X_1X_2 + X_2X_4 \\ X_3X_1 + X_4X_3 & X_3X_2 + X_4^2 \end{pmatrix} = 0 \tag{1.4.3}$$

$$\mathcal{F}_{m+1} = \mathcal{F}_{m-1}^{\perp} = \text{colspan} \begin{pmatrix} Y \\ I_n \end{pmatrix}, \text{ where } Y = \begin{pmatrix} -J_{n-2} & \\ & H_2 \end{pmatrix} X^t \begin{pmatrix} J_{n-2} & \\ & H_2 \end{pmatrix},$$

$\pi \otimes 1$-stability on $\mathcal{F}_{m+1}$ follows from $\pi \otimes 1$-stability on $\mathcal{F}_{m-1}$.

The condition $\mathcal{F}_{m-1}$ is mapped into $\mathcal{F}_{m+1}$ is equivalent to $\mathcal{F}_{m-1}^t A_{m-1}^t M \mathcal{F}_{m-1} = 0$, i.e.

$$\begin{pmatrix} -J_{n-2}X_1 + X_3^tH_2X_3 + X_1^tJ_{n-2} & -J_{n-2}X_2 + X_3^tH_2X_4 \\ X_4^tH_2X_3 + X_2^tJ_{n-2} & X_4^tH_2X_4 \end{pmatrix} = 0 \tag{1.4.4}$$

and that $\mathcal{F}_{m+1}$ is mapped into $\mathcal{F}_{n+m-1}$ is equivalent to $\mathcal{F}_{m+1}^t M A_{m+1} \mathcal{F}_{m+1} = 0$, i.e.

$$\begin{pmatrix} X_1J_{n-2}X_1^t & X_1J_{n-2}X_3^t - X_2H_2 \\ X_3J_{n-2}X_1^t - X_2H_2 & X_3J_{n-2}X_3^t - X_4H_2 + H_2X_4^t \end{pmatrix} \tag{1.4.5}$$

wedge condition is equivalent to

$$\Lambda^2 X = 0. \tag{1.4.6}$$

Then R-points satisfying (LM1)-(LM4) and (LM6) are represented by X such that (1.4.3)-(1.4.6) hold.

1.4.2 A basis for im$[W(\Lambda_{m-1})_{-1}^{n-1,1} \otimes_{\mathcal{O}_F} R \longrightarrow W(\Lambda_{m-1}) \otimes_{\mathcal{O}_F} R]$

For computations about strengthened spin condition, take $g_1, \dots, g_{2n}$ to be the F-basis

$$e_1 \otimes 1 - \pi e_1 \otimes \pi^{-1}, \cdots, e_n \otimes 1 - \pi e_n \otimes \pi^{-1}, \frac{e_1 \otimes 1 + \pi e_1 \otimes \pi^{-1}}{2}, \cdots, \frac{e_n \otimes 1 + \pi e_n \otimes \pi^{-1}}{2} \tag{1.4.7}$$

for V, since it splits and separates $V_{-\pi}$ and V_{π} at the same time, and they induce the same $W_{\pm 1}$ as (1.2.2) because the transformation between them has determinant 1, then

$$W_{\pm 1} = \mathrm{span}_F\{g_S \pm \mathrm{sgn}(\sigma_S) g_{S^\perp} | \# S = n\}$$

where $g_S \in W$ is defined with respect to the basis (1.4.7), and

$$W_{-1}^{n-1,1} = \mathrm{span}_F\{g_S - \mathrm{sgn}(\sigma_S) g_{S^\perp} | S = \{1, \dots, \hat{j}, \dots n, n+i\}, 1 \leq i, j \leq n\}$$

where the hat means the element is omitted, and

$$W(\Lambda_{m-1}) = \mathrm{span}_{\mathcal{O}_F}\{e_S | \# S = n\}$$

where $e_S \in W$ is defined with respect to basis (1.3.1). $\begin{pmatrix} X \\ I_n \end{pmatrix}$ is with respect to (1.4.1), a reordering of (1.3.1), therefore naturally we hope to find an $\mathcal{O}_F$-basis for $W(\Lambda_{m-1})_{-1}^{n-1,1}$ in terms of e_S and from this an R-basis for $\mathrm{im}[W(\Lambda_{m-1})_{-1}^{n-1,1} \otimes_{\mathcal{O}_F} R \longrightarrow W(\Lambda_{m-1}) \otimes_{\mathcal{O}_F} R]$, then apply the strengthened spin condition easily to entries of X. And in fact it is easier: since $\pi = 0$ in R, only the lower order terms in $g_S - \mathrm{sgn}(\sigma_S) g_{S^\perp}$ matter.

Definition 1. *Let $w = \sum_S c_S e_S \in W$, the worst term of w is*

$$WT(w) := \sum c_S e_S,$$

the sum taken over S such that $ord_\pi(c_S) \leq ord_\pi(c'_S)$ for any S'.

Suppose $S = \{1, \dots, \hat{j}, \dots n, n+i\}$, $S^\perp = \{1, \dots, \hat{i^\vee}, \dots n, n+j^\vee\}$. $\mathrm{sgn}(\sigma_S) = (-1)^{i+j}$.

Lemma 1. *1. When $S = \{1, \dots, \hat{i}, \dots n, n+i\}$,*

(1) for $i \leq m-1$,

$$WT(g_S) = \frac{(-1)^{m+i+1}}{2} \pi^{-(m+1)} e_{\{n+1, \dots, 2n\}}$$

(2) for $i > m-1$,

$$WT(g_S) = \frac{(-1)^{m+i}}{2} \pi^{-(m+1)} e_{\{n+1, \dots, 2n\}}$$

2. When $S = \{1, \dots, \hat{j}, \dots n, n+i\}, j \neq i$, *(3) for $i, j \leq m-1$,*

$$WT(g_S) = (-1)^m \pi^{-m} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$

(4) for $i \leq m-1 < j$,

$$WT(g_S) = (-1)^{m+1} \pi^{-(m-1)} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$

(5) for $j \leq m-1 < i$,

$$WT(g_S) = (-1)^m \pi^{-(m+1)} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$

(6) for $i, j > m-1$,

$$WT(g_S) = (-1)^{m+1} \pi^{-m} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$

Proof. The lower order terms in $g_i's$ are the underlined ones

$$\underline{e_1 \otimes 1} - \pi e_1 \otimes \pi^{-1}, \cdots, \underline{e_{m-1} \otimes 1} - \pi e_{m-1} \otimes \pi^{-1}, e_m \otimes 1 \underline{- \pi e_m \otimes \pi^{-1}}, \cdots, e_n \otimes 1 \underline{- \pi e_n \otimes \pi^{-1}},$$

$$\underline{\frac{1}{2} e_1 \otimes 1} + \frac{1}{2} \pi e_1 \otimes \pi^{-1}, \cdots, \underline{\frac{1}{2} e_{m-1} \otimes 1} + \frac{1}{2} \pi e_{m-1} \otimes \pi^{-1}, \frac{1}{2} e_m \otimes 1 + \underline{\frac{1}{2} \pi e_m \otimes \pi^{-1}}, \cdots, \frac{1}{2} e_n \otimes 1 + \underline{\frac{1}{2} \pi e_n \otimes \pi^{-1}},$$

and

$$g_i \wedge g_{n+i} = (e_i \otimes 1) \wedge (\pi e_i \otimes \pi^{-1}) = (-\pi e_i \otimes \pi^{-1}) \wedge (e_i \otimes 1),$$

then in each case

$$\begin{aligned}(1) WT(g_S) =& (e_1 \otimes 1) \wedge \cdots \wedge (\widehat{e_i \otimes 1}) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\ & \wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (\frac{1}{2} e_i \otimes 1).\end{aligned}$$

$$\begin{aligned}(2) WT(g_S) =& (e_1 \otimes 1) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\ & \wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi \widehat{e_i \otimes \pi}^{-1}) \ldots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (\frac{1}{2} \pi e_i \otimes \pi^{-1}).\end{aligned}$$

$$
\begin{aligned}
(3)WT(g_S) =&(e_1 \otimes 1) \wedge \cdots \wedge (\widehat{e_j \otimes 1}) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\
&\wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (\pi e_i \otimes \pi^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
(4)WT(g_S) =&(e_1 \otimes 1) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\
&\wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi \widehat{e_j \otimes \pi^{-1}}) \ldots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (\pi e_i \otimes \pi^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
(5)WT(g_S) =&(e_1 \otimes 1) \wedge \cdots \wedge (\widehat{e_j \otimes 1}) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\
&\wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (e_i \otimes 1).
\end{aligned}
$$

$$
\begin{aligned}
(6)WT(g_S) =&(e_1 \otimes 1) \wedge \cdots \wedge (e_{m-1} \otimes 1) \\
&\wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \wedge (-\pi \widehat{e_j \otimes \pi^{-1}}) \ldots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge (e_i \otimes 1).
\end{aligned}
$$

$\square$

Now we are ready to compute the worst terms of $g_S - \mathrm{sgn}(\sigma_S) g_{S^\perp}$ for all $S = \{i, \ldots, \hat{j}, \ldots, n, n+i\}$, suppose $i \leq j^\vee$ by symmetry.

Lemma 2. *1. When $S = S^\perp, S = \{1, \ldots, \hat{i^\vee}, \ldots, n, n+i\}, sgn(\sigma_S) = -1$,*

(1) for $i \leq m-1$, then $i^\vee > m-1$,

$$WT(g_S - sgn(\sigma_S) g_{S^\perp}) = 2(-1)^{m+1} \pi^{-(m-1)} e_{\{i, n+1, \ldots, \hat{i^*}, \ldots, 2n\}};$$

(2) for $m \leq i \leq m+1$, then $i^\vee > m-1$,

$$WT(g_S - sgn(\sigma_S) g_{S^\perp}) = 2(-1)^{m+1} \pi^{-m} e_{\{i, n+1, \ldots, \hat{i^*}, \ldots, 2n\}};$$

(3) for $i \geq m+2$, then $i^\vee \leq m-1$,

$$WT(g_S - sgn(\sigma_S) g_{S^\perp}) = 2(-1)^{m} \pi^{-(m+1)} e_{\{i, n+1, \ldots, \hat{i^*}, \ldots, 2n\}};$$

2. When $S = \{1, \ldots, \hat{i}, \ldots, n, n+i\}, sgn(\sigma_S) = 1$, *(4) for $i \le m-1$, then $i^\vee > m-1$,*

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{m+1}\pi^{-m}(e_{\{i,n+1,\dots,\widehat{n+i},\dots,2n\}} + e_{\{i^\vee,n+1,\dots,\hat{i^*},\dots,2n\}});$$

(5) for $i = m$, then $i^\vee = m+1$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = \pi^{-(m+1)}e_{\{n+1,\dots,2n\}};$$

3. When $S = \{1,\dots,\hat{j},\dots,n,n+i\}, j \neq i, i < j^\vee$,

(6) for $i < j^\vee \le m-1$, then $i^\vee > j > m-1$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{m+1}\pi^{-(m-1)}(e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j+1}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}});$$

(7) for $i \le m-1 < j^\vee < m+2$, then $i^\vee > j > m-1$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{i+j+m}\pi^{-m}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}};$$

(8) for $i \le m-1, j^\vee \ge m+2$, then $j \le m-1 < i^\vee$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{m}\pi^{-m}(e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}});$$

(9) for $m-1 < i < m+2 \le j^\vee$, then $j \le m-1 < i^\vee$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{m}\pi^{-(m+1)}e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}};$$

(10) for $m+2 \le i < j^\vee$, then $j < i^\vee \le m-1$,

$$WT(g_S - sgn(\sigma_S)g_{S^\perp}) = (-1)^{m}\pi^{-(m+1)}(e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j+1}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}}).$$

Proof. (1)-(3) are obvious.

(4)

$$WT(g_S) = \frac{(-1)^{m+i+1}}{2}\pi^{-(m+1)}e_{\{n+1,\dots,2n\}}$$

and

$$WT(g_{S^\perp}) = \frac{(-1)^{m+i^\vee}}{2}\pi^{-(m+1)}e_{\{n+1,\dots,2n\}} = WT(g_S)$$

cancel out, so other terms in g_S and $g_{S^\perp}$ need to be analyzed.

$$\begin{aligned}
g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp} =& g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_{i^\vee} \wedge \cdots \wedge g_n \wedge g_{n+i} - g_1 \wedge \cdots \wedge g_i \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n \wedge g_{n+i^\vee} \\
=&(-1)^{n-i^\vee}(g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n) \wedge (g_{i^\vee} \wedge g_{n+i}) \\
&- (-1)^{n-i-1}(g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n) \wedge (g_i \wedge g_{n+i^\vee}) \\
=&(-1)^i(g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n) \wedge (g_i \wedge g_{n+i^\vee} - g_{i^\vee} \wedge g_{n+i}) \\
=&(-1)^i(g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n)[(e_i \otimes 1) \wedge (e_{i^\vee} \otimes 1) - (\pi^{-1} e_i \otimes 1) \wedge (\pi e_{i^\vee} \otimes 1)],
\end{aligned}$$

so

$$\begin{aligned}
WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) =&(-1)^i (e_1 \otimes 1) \wedge \cdots \wedge \widehat{(e_i \otimes 1)} \wedge \cdots \wedge (e_{m-1} \otimes 1) \wedge (-\pi e_m \otimes \pi^{-1}) \wedge \cdots \\
&\wedge (-\pi \widehat{e_{i^\vee} \otimes} \pi^{-1}) \ldots \wedge (-\pi e_n \otimes \pi^{-1}) \wedge [(e_i \otimes 1) \wedge (e_{i^\vee} \otimes 1) - (\pi^{-1} e_i \otimes 1) \wedge (\pi e_{i^\vee} \otimes 1)] \\
=&(-1)^m \pi^{-m}(e_{\{i,n+1,\ldots,\widehat{n+i},\ldots,2n\}} + e_{\{i^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})
\end{aligned}$$

(5)

$$\begin{aligned}
WT(g_S) &= \frac{1}{2}\pi^{-(m+1)} e_{\{n+1,\ldots,2n\}} \\
WT(g_{S^\perp}) &= -\frac{1}{2}\pi^{-(m+1)} e_{\{n+1,\ldots,2n\}}
\end{aligned}$$

so

$$WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) = \pi^{-(m+1)} e_{\{n+1,\ldots,2n\}}$$

(6)

$$\begin{aligned}
WT(g_S) &= (-1)^{m+1}\pi^{-(m-1)} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} \\
WT(g_{S^\perp}) &= (-1)^{m+1}\pi^{-(m-1)} e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}
\end{aligned}$$

so

$$WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) = (-1)^{m+1}\pi^{-(m-1)}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j+1} e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})$$

(7)

$$WT(g_S) = (-1)^{m+1}\pi^{-(m-1)}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$
$$WT(g_{S^\perp}) = (-1)^{m+1}\pi^{-m}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$$

so

$$WT(g_S - \mathrm{sgn}(\sigma_S)g_{S^\perp}) = (-1)^{i+j+m}\pi^{-m}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$$

(8)

$$WT(g_S) = (-1)^{m}\pi^{-m}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$
$$WT(g_{S^\perp}) = (-1)^{m+1}\pi^{-m}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$$

so

$$WT(g_S - \mathrm{sgn}(\sigma_S)g_{S^\perp}) = (-1)^{m}\pi^{-m}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})$$

(9)

$$WT(g_S) = (-1)^{m}\pi^{-(m+1)}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$
$$WT(g_{S^\perp}) = (-1)^{m+1}\pi^{-m}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$$

so

$$WT(g_S - \mathrm{sgn}(\sigma_S)g_{S^\perp}) = (-1)^{m}\pi^{-(m+1)}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$

(10)

$$WT(g_S) = (-1)^{m}\pi^{-(m+1)}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$$
$$WT(g_{S^\perp}) = (-1)^{m}\pi^{-(m+1)}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$$

so

$$WT(g_S - \mathrm{sgn}(\sigma_S)g_{S^\perp}) = (-1)^m\pi^{-(m+1)}(e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j+1}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}})$$

$\square$

Using the same argument as in[11], one can verify

Proposition 2. *$im[W(\Lambda_{m-1})^{n-1,1}_{-1} \otimes_{\mathcal{O}_F} R \longrightarrow W(\Lambda_{m-1}) \otimes_{\mathcal{O}_F} R]$ is a free R-module on the basis:*

$$e_{\{n+1,\dots,2n\}}; \tag{1.4.8}$$

$$e_{\{i,n+1,\dots,\hat{i^*},\dots,2n\}},\ \textit{for } \forall i; \tag{1.4.9}$$

$$e_{\{i,n+1,\dots,\widehat{n+i},\dots,2n\}} + e_{\{i^\vee,n+1,\dots,\hat{i^*},\dots,2n\}},\ \textit{for } i \le m-1; \tag{1.4.10}$$

$$e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j+1}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}},\ \textit{for } i < j^\vee \le m-1, m+2 \le i < j^\vee; \tag{1.4.11}$$

$$e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} + (-1)^{i+j}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}},\ \textit{for } i \le m-1, m+2 \le j^\vee, i \ne j; \tag{1.4.12}$$

$$e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}},\ \textit{for } m \le i \le m+1, j^\vee \le m-1 \textit{ or } m+2 \le j^\vee. \tag{1.4.13}$$

1.4.3 The local chart

In this subsection we gather the restrictions on X given by strengthened spin condition through the basis in Proposition 2, and combine with (1.4.3)-(1.4.6) to conclude the local chart around the worst point.

Let $\bigwedge^n_R \mathcal{F}_{m-1} = \sum c_S e_S \in W(\Lambda_{m-1}) \otimes_{\mathcal{O}_F} R$. First notice that the $e_S's$ not showing up in the basis are:

$$e_S, \text{ for } \#(S \cap \{1,\dots,n\}) \ge 2; \tag{1.4.14}$$

$$e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}, \text{ for } m \le j \le m+1, j \ne i^\vee. \tag{1.4.15}$$

In other words, $c_S = 0$ for such S. And

(1.4.14) means $\Lambda^2 X = 0$.

(1.4.15) means $X_2 = 0$ and $(X_4)_{11} = (X_4)_{22} = 0$.

Then look at the basis elements which are linear combinations of more than one e_S.

(1.4.10) means $c_{\{i,n+1,\dots,\widehat{n+i},\dots,2n\}} = c_{\{i^\vee,n+1,\dots,\hat{i^*},\dots,2n\}}$, for $i \le m-1$. Since

$e_{\{i,n+1,\dots,\widehat{n+i},\dots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$(\pi e_{m+2}\otimes 1)\wedge\cdots\wedge(\pi e_n\otimes 1)\wedge(e_1\otimes 1)\wedge\cdots\wedge(\pi^{-1}e_i\otimes d_{ii})\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)$$
$$=(-1)^{m+i}d_{ii}e_{\{i,n+1,\dots,\widehat{n+i},\dots,2n\}},$$

$e_{\{i^\vee,n+1,\dots,\hat{i^*},\dots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$(\pi e_{m+2}\otimes 1)\wedge\cdots\wedge(e_{i^\vee}\otimes a_{m-i,m-i})\wedge\cdots\wedge(\pi e_n\otimes 1)\wedge(e_1\otimes 1)\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)$$
$$=(-1)^{m+i+1}a_{m-i,m-i}e_{\{i^\vee,n+1,\dots,\hat{i^*},\dots,2n\}},$$

we get

$$(-1)^{m+i}d_{ii}=(-1)^{m+i+1}a_{m-i,m-i},\ \text{i.e. } d_{ii}=-a_{m-i,m-i}.$$

(1.4.12) means $c_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}} = (-1)^{i+j}c_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}}$, for $i\le m-1, j^\vee\ge m+2, i\neq j$;. Since

$e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$\begin{aligned}(\pi e_{m+2}\otimes 1)&\wedge\cdots\wedge(\pi e_n\otimes 1)\wedge(e_1\otimes 1)\wedge\cdots\wedge(e_{j-1}\otimes 1)\\ &\wedge(\pi^{-1}e_i\otimes d_{ij})\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)\\ &=(-1)^{m+j}d_{ij}e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}},\end{aligned}$$

$e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$\begin{aligned}(\pi e_{m+2}\otimes 1)&\wedge\cdots\wedge(\pi e_{i^\vee-1}\otimes 1)\wedge(e_{j^\vee}\otimes a_{m-j,m-i})\wedge\cdots\wedge(\pi e_n\otimes 1)\\ &\wedge(e_1\otimes 1)\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)\\ &=(-1)^{m+i+1}a_{m-j,m-i}e_{\{j^\vee,n+1,\dots,\hat{i^*},\dots,2n\}},\end{aligned}$$

it follows

$$(-1)^{m+j}d_{ij}=(-1)^{m+i+1}a_{m-j,m-i},\ \text{i.e. } d_{ij}=-a_{m-j,m-i}.$$

Together with $d_{ii}=-a_{m-i,m-i}$ we get

$$D=-A^{ad},$$

where $A^{ad} = H_{m-1}A^tH_{m-1}$.

(1.4.11) means $c_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} = (-1)^{i+j+1}c_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$, for $i < j^\vee \le m-1$ or $j^\vee > i \ge m+2$.

For $i < j^\vee \le m-1$, $e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$\begin{aligned}(\pi e_{m+2}\otimes 1)\wedge\cdots\wedge(\pi e_{j-1}\otimes 1)\wedge(\pi^{-1}e_i\otimes c_{i,j-m-1})\wedge\cdots\wedge(\pi e_n\otimes 1)&\\ \wedge(e_1\otimes 1)\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)&\\ =(-1)^{m+j}c_{i,j-m-1}e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}&\end{aligned}$$

$e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}$ term in $\Lambda^n_R\mathcal{F}_{m-1}$ is

$$\begin{aligned}(\pi e_{m+2}\otimes 1)\wedge\cdots\wedge(\pi e_{i^\vee-1}\otimes 1)\wedge(e_{j^\vee}\otimes c_{j^\vee,m-i}\pi^{-1})\wedge\cdots\wedge(\pi e_n\otimes 1)&\\ \wedge(e_1\otimes 1)\wedge\cdots\wedge(e_{m-1}\otimes 1)\wedge(\pi e_m\otimes 1)\wedge(\pi e_{m+1}\otimes 1)&\\ =(-1)^{m+i+1}c_{j^\vee,m-i}e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}&\end{aligned}$$

so $(-1)^{m+j}c_{i,j-m-1} = (-1)^{i+j+1}(-1)^{m+i+1}c_{j^\vee,m-i}$, i.e. $c_{i,j-m-1} = c_{m-(j-m-1),m-i}$.

hence

$$C = C^{ad}.$$

For $j^\vee > i \ge m+2$, we get

$$B = B^{ad}$$

similarly.

(LM7) on $\mathcal{F}_{m+1}$ provides the same restrictions on X if $\mathcal{F}_{m+1} = \mathcal{F}_{m-1}^\perp$.

To sum up, X satisfies

$$\Lambda^2 X = 0; \tag{1.4.16}$$

$$X_2 = 0, (X_4)_{11} = (X_4)_{22} = 0; \tag{1.4.17}$$

$$D = -A^{ad}, B = B^{ad}, C = C^{ad}; \tag{1.4.18}$$

$$X_1^2 = 0, X_3X_1 + X_4X_3 = 0, X_4^2 = 0; \tag{1.4.19}$$

$$J_{n-2}X_1 - X_1^tJ_{n-2} = X_3^tH_2X_3, X_4^tH_2X_3 = 0, X_4^tH_2X_4 = 0. \tag{1.4.20}$$

$$X_1J_{n-2}X_1^t = 0, X_1J_{n-2}X_3^t = 0, X_3J_{n-2}X_3^t = 0. \tag{1.4.21}$$

Write

$$X_4 = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} E & F \end{pmatrix}$$

, where E and F are of size $2 \times (m-1)$, then $X_4^2 = 0$ gives $x_1 x_2 = 0$, which also guarantees $X_4^t H_2 X_4 = 0$.

$X_4^t H_2 X_3 = 0$ gives

$$\begin{pmatrix} x_2 & 0 \\ 0 & x_1 \end{pmatrix} X_3 = 0,$$

then $X_4 X_3 = 0$, and thus $X_3 X_1 + X_4 X_3 = X_3 X_1 = 0$. Now

$$J_{n-2} X_1 = \begin{pmatrix} HC & HD \\ -HA & -HB \end{pmatrix} = \begin{pmatrix} C^t H & -A^t H \\ D^t H & -B^t H \end{pmatrix} = -X_1^t J_{n-2}.$$

so

$$X_1 = -\frac{1}{2} J_{n-2} X_3^t H_2 X_3,$$

by (1.4.20), i.e.

$$A = -\frac{1}{2} F^{ad} E, B = -\frac{1}{2} F^{ad} F, C = \frac{1}{2} E^{ad} E, D = \frac{1}{2} E^{ad} F,$$

such A, B, C, D also satisfy symmetric conditions (1.4.18).

Now $X_3 X_1 = 0$ and $X_3 J_{n-2} X_3^t = 0$ can be deduced from $\Lambda^2 X_3 = 0$:

$$X_3 X_1 = \begin{pmatrix} E & F \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (FH_{m-1}E^t - EH_{m-1}F^t) H_2 E & (FH_{m-1}E^t - EH_{m-1}F^t) H_2 F \end{pmatrix},$$

$$X_3 J_{n-2} X_3^t = EH_{m-1}F^t - FH_{m-1}E^t,$$

and $\Lambda^2 X_3 = 0$ implies

$$FH_{m-1}E^t - EH_{m-1}F^t = 0.$$

Since $X_1 = -\frac{1}{2} J_{n-2} X_3^t H_2 X_3$, $X_1^2 = 0$ follows from $X_3 X_1 = 0$, and $X_1 J_{n-2} X_1^t = 0, X_1 J_{n-2} X_3^t = 0$ follow from $X_3 J_{n-2} X_3^t = 0$. And wedge condition can be reduced to $\Lambda^2 \begin{pmatrix} X_3 & X_4 \end{pmatrix} = 0$.

Therefore, (1.4.16)-(1.4.21) is equivalent to X_3 and x_1, x_2 satisfying

$$x_1 x_2 = 0; \tag{1.4.22}$$

$$\begin{pmatrix} x_2 & 0 \\ 0 & x_1 \end{pmatrix} X_3 = 0; \tag{1.4.23}$$

$$\Lambda^2 X_3 = 0. \tag{1.4.24}$$

Write $Y = (X_4 \ X_3)$, then the closed subscheme cut out by all the conditions in the affine open around the worst point is Spec $k[Y]/(\Lambda^2 Y + (y_{11}, y_{22})) = \mathrm{Spec}(k[Y]/\Lambda^2 Y)/(\bar{y}_{11}, \bar{y}_{22})$.

Remark 1. *We have also seen in this case the strengthened spin condition implies the wedge condition.*

1.4.4 Proof of Proposition 1

Lemma 3. *Let k be a field and X an $m \times n$ matrix of indeterrminate over k, t a positive integer $\leq \min\{m, n\}$, $I_t(X)$ the ideal generated by the t-minors of X.*

Then the standard bitableaux $\gamma_1 \ldots \gamma_u$ such that $|\gamma_1| \geq t$ form a k-basis of $I_t(X)$, and the images of standard bitableaux $\delta_1 \ldots \delta_v$ such that $|\delta_1| \leq t-1$ form a k-basis of $k[X]/I_t(X)$.

proof of Proposition 1. by Lemma 3, $k[Y]/\Lambda^2 Y$ has a k-basis

$$\mathcal{B} = \{\prod_{i=1}^{m} \bar{y}_{a_i b_i} : 1 \leq a_1 \leq \ldots \leq a_m \leq 2, 1 \leq b_1 \leq \ldots \leq b_m \leq n\}$$

$k[Y]$ is a $\mathbb{Z}^n \oplus \mathbb{Z}^2$-graded algebra if we give y_{ij} the vector bidegree $e_j \oplus f_i$, then all minors and bitableaux are homogeneous with respect to this grading, and among all the monomials of any same bidegree there is exactly one standard bitableau, i.e. the grading gives a bijection

$$\mathcal{B} \longrightarrow \{(x_1, \ldots, x_n, y_1, y_2) \in \mathbb{Z}_{\geq 0}^{n+2} : \Sigma x_i = \Sigma y_i\}.$$

In $k[Y]/\Lambda^2 Y$, all homogeneous monomials are equal (to the standard one).

So multiplication of monomials is just addition of bidegrees: the product of some standard monomials is the standard monomial whose bidegree is the sum of the bidegrees.

$\mathcal{B}$ is closed under multiplication, so

$$\Sigma_{\gamma \in \mathcal{B}} c_\gamma \gamma \in (\bar{y}_{11}, \bar{y}_{22}) \text{ iff each } \gamma \in (\bar{y}_{11}, \bar{y}_{22}).$$

Using the lexicographical order on $\mathbb{Z}^{n+2}$, every $f \in k[Y]/\Lambda^2 Y$ has a leading term $\gamma \in \mathcal{B}$, then f^2 has leading term $c_\gamma^2 \gamma^2$.

If $f \in (\bar{y}_{11}, \bar{y}_{22})$, $\gamma^2 \in (\bar{y}_{11}, \bar{y}_{22})$, γ^2 has one of the form

$$\bar{y}_{11}, \dots, \bar{y}_{22}, \dots, \bar{y}_{12}, \dots, \bar{y}_{2j}, \dots.$$

But $(\Pi \bar{y}_{a_i b_i})^2 = \Pi \bar{y}_{a_i b_i}^2$, so in each case γ has the same form, $\gamma \in (\bar{y}_{11}, \bar{y}_{22})$.

Then $(f - c_\gamma \gamma)^2 \in (\bar{y}_{11}, \bar{y}_{22})$, by induction $f \in (\bar{y}_{11}, \bar{y}_{22})$.

$(k[Y]/\Lambda^2 Y)/(\bar{y}_{11}, \bar{y}_{22})$ is reduced. So is the whole special fiber by the discussion in §3. $\square$

Remark 2. *Note that if (LM7) is replaced by the spin condition* (1.2.3)*, $M^{spin}_{\{m-1\}}$ is not flat, cf. [8]Remark 9.14.*

1.5 The singularity of $M^{\mathbf{loc}}_{\{m-1,m\}}$

In this section we look deeper into the special fiber and complete the proof of 2.

proof of Theorem 2. The closed subscheme cut out by (1.4.22)-(1.4.24) has three irreducible components:

by (1.4.22) and (1.4.23), either x_1, x_2 are both 0 or one row of Y is 0. In the latter case, $\wedge^2 X_3 = 0$ becomes trivial. Thus

$$(M^{\mathrm{loc}}_{\{m-1,m\}})_s \cap U = Z_1 \cup Z_2 \cup Z_3,$$

where

$$Z_1 = \mathrm{Spec}\ k[X_3]/\wedge^2 X_3,$$

$$Z_2 = \mathrm{Spec}\ k[(X_3)_{11}, \dots, (X_3)_{1,n-2}, x_1],$$

$$Z_3 = \mathrm{Spec}\ k[(X_3)_{21}, \dots, (X_3)_{2,n-2}, x_2].$$

They are all normal, the special fiber is smooth except at the worst point z_0, and consequently the local model is regular except at z_0.

In particular it does not have semi-stable reduction, which can also be seen from the fact that $Z_2 \cap Z_3$ is the single point z_0, failing to have normal crossing. $\square$

2

A Modified Moduli Problem for Resolution of Singularity

2.1 Introduction

In Chapter 1 we also see that the local model does have singularity, although mild. To resolve the singularities, we find a moduli description for the blow-up along the singular locus. With the idea of splitting models in [7], we introduce a similar construction $\tilde{M}_I$ and show the following result.

Theorem 3. $\tilde{M}_{\{m-1,m\}}$ *is flat over* Spec $\mathcal{O}_E$ and regular of dimension n. The generic fiber is smooth of dimension $n-1$, and the special fiber is reduced and consists of 4 irreducible smooth components of dimension $n-1$ with normal crossings. $\tilde{M}_{\{m-1,m\}} \to M_{\{m-1,m\}}$ is a blow-up along one point.

2.2 Splitting local model

We turn to the construction of splitting model for a potential fix. In the setting of §2, the splitting local model $\tilde{M}_I$ is the moduli space on the category of $\mathcal{O}_E$-algebras which associates each $\mathcal{O}_E$-algebra R with the families

$$(\mathcal{F}_i^0 \subset \mathcal{F}_i \subset \Lambda_i \otimes_{\mathcal{O}_{F_0}} R)_{i \in \pm I + n\mathbb{Z}}$$

such that for each such i, $\mathcal{F}_i$ satisfies (LM1)-(LM7), and

(LM1') $\mathcal{F}_i^0$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ submodule of $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R$ and a direct summand of rank s;

(LM2') For each $i < j$, the morphism $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R \to \Lambda_j \otimes_{\mathcal{O}_{F_0}} R$ maps $\mathcal{F}_i^0$ into $\mathcal{F}_j^0$:

$$\begin{array}{ccc} \Lambda_i \otimes_{\mathcal{O}_{F_0}} R & \longrightarrow & \Lambda_j \otimes_{\mathcal{O}_{F_0}} R \\ \cup & & \cup \\ \mathcal{F}_i^0 & \longrightarrow & \mathcal{F}_j^0 \end{array}$$

(LM3') The isomorphism $\Lambda_i \otimes_{\mathcal{O}_{F_0}} R \longrightarrow \Lambda_{i-n} \otimes_{\mathcal{O}_{F_0}} R$ induced by $\Lambda_i \xrightarrow{\pi\otimes 1} \Lambda_{i-n}$ identifies $\mathcal{F}_i^0$ with $\mathcal{F}_{i-n}^0$;

(LM4') $(\pi \otimes 1 + 1 \otimes \pi)(\mathcal{F}_i) \subset \mathcal{F}_i^0$;

(LM5') $(\pi \otimes 1 - 1 \otimes \pi)(\mathcal{F}_i^0) = 0$.

This new moduli space is again represented by a projective $\mathcal{O}_E$-scheme, namely a closed subscheme of a product of finitely many $\mathrm{Gr}(1, \Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E)$ and $\mathrm{Gr}(n, \Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E), 0 \leq i < n$. The projection $\tilde{M}_I \to M_I$ is an isomorphism on the level of generic fibers, in which scenario $\mathcal{F}_i^0$ is uniquely determined by $\mathcal{F}_i$.

Now we return to the case $n = 2m \geq 4, I = \{m-1, m\}, (r, s) = (n-1, 1)$. Similar to Proposition 4.3 in [3], more can be said:

Proposition 3. *$\pi : \tilde{M}_{\{m-1,m\}} \setminus \pi^{-1}(z_0) \to M_{\{m-1,m\}} \setminus \{z_0\}$ is isomorphism.*

2.3 Semi-stable reduction of $\tilde{M}_{\{m-1,m\}}$

proof of Theorem 3. The generic fiber is $\mathbb{P}^{n-1}$ since the generic fiber of M_I is $\mathrm{Gr}(r, n)$, and by the above discussion, $\tilde{M}_I$ and M_I have isomorphic generic fibers. It is smooth of dimension $n-1$.

With Theorem 2 and Proposition 3, proving the statements about $\tilde{M}_{\{m-1,m\}}$ reduces to a neighborhood of $\pi^{-1}(z_0)$.

First we work in the special fiber. Denote π_1 as the map induced by π on the special fiber.

To analyze the irreducible components of $(\tilde{M}_{\{m-1,m\}})_s$, it suffices to work in the neighborhood of $\pi_1^{-1}(z_0) : \pi_1^{-1}(U)$. Here

$$\mathcal{F}_{m-1} = \mathrm{colspan}\begin{pmatrix} X \\ I_n \end{pmatrix}, \text{ where } X = \begin{pmatrix} X_1 & \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} A & B & & \\ C & D & & \\ E_1 & F_1 & & x_1 \\ E_2 & F_2 & x_2 & \end{pmatrix}$$

with entries in a k-algebra R, then

$$\mathcal{F}_m = \operatorname{colspan}\begin{pmatrix} A & B & -F_1^{\mathrm{ad}} & \\ C & D & E_1^{\mathrm{ad}} & \\ & & x_1 & \\ E_2 & F_2 & 1 & \\ I_{m-1} & & & \\ & I_{m-1} & & \\ E_1 & F_1 & & x_1 \\ & & & 1 \end{pmatrix},$$

following the results in [8],

$$\mathcal{F}_{m+1} = \mathcal{F}_{m-1}^{\perp} = \operatorname{colspan}\begin{pmatrix} Y \\ I_n \end{pmatrix}, \text{ where } Y = \begin{pmatrix} -A & -B & -F_2^{\mathrm{ad}} & -F_1^{\mathrm{ad}} \\ -C & -D & E_2^{\mathrm{ad}} & E_1^{\mathrm{ad}} \\ & & & x_1 \\ & & x_2 & \end{pmatrix}$$

and hence $\mathcal{F}_m^0 = \operatorname{colspan}\begin{pmatrix} -F_1^{\mathrm{ad}} \\ E_1^{\mathrm{ad}} \\ x_1 \\ 1 \end{pmatrix}$ with respect to the basis

$$\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_m \otimes 1, \pi e_{m+1} \otimes 1\}.$$

Let $\tilde{Z}_0 = \pi_1^{-1}(z_0)$, which represents the pairs $(\mathcal{F}_{m-1}^0, \mathcal{F}_{m+1}^0)$ such that $\mathcal{F}_{m-1}^0 \subset (\pi \otimes 1)(\Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} R), \mathcal{F}_{m+1}^0 \subset (\pi \otimes 1)(\Lambda_{m+1} \otimes_{\mathcal{O}_{F_0}} R)$, and $\mathcal{F}_{m-1}^0, \mathcal{F}_m^0 = \operatorname{span}_R\{\pi e_{m+1} \otimes 1\}, \mathcal{F}_{m+1}^0$ all land into where they should under the maps T_{ij} induced by inclusions, i.e.

$$T_{m-1,m}(\mathcal{F}_{m-1}^0) \subset \mathcal{F}_m^0, \text{ and } T_{m+1,n+m-1}(\mathcal{F}_{m+1}^0) \subset \mathcal{F}_{n+m-1}^0$$

describing $\tilde{Z}_0$ as the blow-up of $\mathbb{P}^{n-1}$ along a $\mathbb{P}^{n-3}$ locus, irreducible and smooth.

Next think about the projection π_0:

$$\pi_1^{-1}(U) \subset \tilde{M}_{\{m-1,m\}} \to \tilde{Z}_0$$
$$(\mathcal{F}_i^0 \subset \mathcal{F}_i) \longmapsto (p(\mathcal{F}_{m-1}), \mathcal{F}_{m+1}^0)$$

where p is the projection to $\mathrm{span}\{\pi e_m \otimes 1, \pi e_{m+1} \otimes 1\}$, it is well-defined because if $(\mathcal{F}_i^0 \subset \mathcal{F}_i) \in \tilde{Z}_0$, it is the identity map, otherwise $\mathcal{F}_{m-1}^0 =$colspan(X), and $X_1 = 0$ if $X_3 = 0$.

Take the closed subfunctor in $\tilde{Z}_0$ where $p(\mathcal{F}_{m-1}^0) \perp (\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0$, it has 3 irreducible components:

$$W_1 = \mathbb{P}^1(\mathrm{span}_k\{\pi e_m \otimes 1, \pi e_{m+1} \otimes 1\}) \times \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1\}) \simeq \mathbb{P}^1 \times \mathbb{P}^{n-3},$$

$$W_2 = \mathbb{P}^1(\mathrm{span}_k\{\pi e_m \otimes 1\}) \times \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1, e_m \otimes 1\}) \simeq \mathbb{P}^{n-2},$$

$$W_3 = \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+1} \otimes 1\}) \times \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1, e_{m+1} \otimes 1\}) \simeq \mathbb{P}^{n-2}.$$

Their intersections are

$$W_1 \cap W_2 = \mathbb{P}^1(\mathrm{span}_k\{\pi e_m \otimes 1\}) \times \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1\}) \simeq \mathbb{P}^{n-3},$$

$$W_1 \cap W_3 = \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+1} \otimes 1\}) \times \mathbb{P}^1(\mathrm{span}_k\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1\}) \simeq \mathbb{P}^{n-3},$$

$W_2 \cap W_3 = \emptyset$.

Let $\tilde{Z}_i = \pi_0^{-1}(W_i)$, for $1 \leq i \leq 3$, then the two factors in W_i provide the column and row spans of $\begin{pmatrix} X_3 & X_4 \end{pmatrix}$. Therefore $\tilde{Z}_i$ is a $\mathbb{A}^1$-bundle of W_i, also irreducible and smooth.

For an R-point in $\pi_1^{-1}(U)$, if $p(\mathcal{F}_{m-1}^0) \not\subset ((\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0)^{\perp}$, necessarily $\mathcal{F}_{m-1}^0 \not\subset ((\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0)^{\perp}$: either

$$\mathcal{F}_{m-1}^0 = p(\mathcal{F}_{m-1}^0)$$

or

$$p'(\mathcal{F}_{m-1}^0) = \mathcal{F}_{m+1}^0 \text{ and } p'(\mathcal{F}_{m-1}^0) \perp (\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0,$$

where p' is the projection to $\mathrm{span}_R\{\pi e_{m+2} \otimes 1, \ldots, \pi e_n \otimes 1, e_1 \otimes 1, \ldots, e_{m-1} \otimes 1\}$.

But $((\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0)^{\perp} \subset \mathcal{F}_{m-1}$ following from $(\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0 \supset \mathcal{F}_{m+1}$ always holds, $\mathcal{F}_{m-1}^0$ and $((\pi \otimes 1)^{-1}\mathcal{F}_{m+1}^0)^{\perp}$ are rank 1 and $n-1$ direct summands of both $\mathcal{F}_{m-1}$ and $(\pi \otimes 1)(\Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} R)$, rank consideration forces

$$\mathcal{F}_{m-1} = \mathcal{F}^0_{m-1} \oplus ((\pi \otimes 1)^{-1}\mathcal{F}^0_{m+1})^{\perp} = (\pi \otimes 1)(\Lambda_{m-1} \otimes_{\mathcal{O}_{F_0}} R)$$

which means this point falls inside $\tilde{Z}_0$.

We have shown

$$\pi_1^{-1}(U) = \tilde{Z}_0 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3,$$

$\tilde{Z}_i$ not containing each other and evidently the 4 components are all irreducible and smooth of dimension $n-1$.

Furthermore, $\tilde{Z}_0 \cap \tilde{Z}_i \simeq W_i$, by virtue of the fact that p restricts to identity in $\tilde{Z}_0$.

$$\tilde{Z}_1 \cap \tilde{Z}_2 \simeq \mathbb{A}^1\text{-bundle of } W_1 \cap W_3 \simeq \mathbb{P}^{n-3},$$

$$\tilde{Z}_1 \cap \tilde{Z}_3 \simeq \mathbb{A}^1\text{-bundle of } W_1 \cap W_3 \simeq \mathbb{P}^{n-3},$$

whereas $\tilde{Z}_2 \cap \tilde{Z}_3 = \emptyset$ since $W_2 \cap W_3 = \emptyset$.

$$\tilde{Z}_0 \cap \tilde{Z}_1 \cap \tilde{Z}_2 \simeq W_1 \cap W_2 \simeq \mathbb{P}^{n-3},$$

$$\tilde{Z}_0 \cap \tilde{Z}_1 \cap \tilde{Z}_3 \simeq W_1 \cap W_3 \simeq \mathbb{P}^{n-3},$$

and all the other intersections are empty.

In particular, the four irreducible components in $\pi_1^{-1}(U)$ and the special fiber have normal crossings.

For the rest of the proof, we go back to the local model $\tilde{M}_{\{m-1,m\}}$ over $\mathcal{O}_F$. In §4 we only study the worst terms of basis elements in $W(\Lambda_{m-1})^{n-1,1}_{-1}$ since the other terms vanish after transferred to the lattice tensored with a k-algebra R. Now we need to dive a little deeper and transform the strengthened spin condition to basis (1.4.1) over $\mathcal{O}_F$-algebra R. Still, take some affine opens U_{ij} of $\prod_{i=m-1}^{m+1}[\mathbb{P}(\Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F) \times \mathrm{Gr}(n, \Lambda_i \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F)$ and write down the equations that define $\tilde{M}_{\{m-1,m\}}$ in a neighborhood of $\pi^{-1}(z_0)$.

Suppose $\mathcal{F}_{m-1}$ =colspan$\begin{pmatrix} X \\ I_n \end{pmatrix}$, where $X = \begin{pmatrix} A & B & L \\ C & D & M \\ E & F & X_4 \end{pmatrix}$, $\mathcal{F}^0_{m-1}$ =colspan$\begin{pmatrix} \pi S \\ S \end{pmatrix}$ and

$\mathcal{F}^0_{m+1}$ =colspan $\begin{pmatrix} \pi T \\ T \end{pmatrix}$ represent the R-points.

$S = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$,and $T = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$, $s_i = t_j = 1$ in U_{ij}.

$\mathcal{F}_{m-1}$ is subject to:

$\pi \otimes 1$-stability: $(\pi \otimes 1) \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} \pi_0 I \\ X \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} T, T = X, X^2 = \pi_0 I_n$,

perpendicularity: $\mathcal{F}^{\mathrm{t}}_{m-1} A^{\mathrm{t}}_{m-1} M \mathcal{F}_{m-1} = 0$, where $A_{m-1} = \begin{pmatrix} I_{n-2} & & & \\ & & & \pi_0 I_2 \\ & & I_{n-1} & \\ & I_2 & & \end{pmatrix}$, i.e.

$$\begin{pmatrix} -J_{n-2} X_1 + X_3^t H_2 X_3 + X_1^t J_{n-2} & -J_{n-2} X_2 + X_3^t H_2 X_4 \\ X_4^t H_2 X_3 + X_2^t J_{n-2} & X_4^t H_2 X_4 - \pi_0 H_2 \end{pmatrix} = 0 \tag{2.3.1}$$

wedge condition: $\wedge^2 \begin{pmatrix} \pi(X + \pi I_n) \\ X + \pi I_n \end{pmatrix} = 0$, and the strengthened spin condition.

When $S = \{1, \ldots, \hat{i}, \ldots, n, n+i\}, i \leq m$,

for $i \leq m-1$,

$$WT(g_S - \mathrm{sgn}(\sigma_S) g_{S^\perp}) = (-1)^{m+1} \pi^{-m} (e_{\{i, n+1, \ldots, \widehat{n+i}, \ldots, 2n\}} + e_{\{i^\vee, n+1, \ldots, \hat{i^*}, \ldots, 2n\}})$$

for $i = m$,

$$g_S - \mathrm{sgn}(\sigma_S) g_{S^\perp} = \pi^{-(m+1)} e_{\{n+1, \ldots, 2n\}} + \sum_{i \neq m, m+1} (-1)^i \pi^{-m} e_{\{i, n+1, \ldots, \widehat{n+i}, \ldots, 2n\}} + \text{other terms}$$

These are the only basis elements involving $e_{\{i, n+1, \ldots, n\hat{+}i, \ldots, 2n\}}$. Compare the coefficients $c_{\{i, n+1, \ldots, n\hat{+}i, \ldots, 2n\}}$ and note that $c_{\{n+1, \ldots, 2n\}} = (-1)^{m+1}$, we conclude

$$(X_4)_{11} = (X_4)_{22} = 0. \tag{2.3.2}$$

again write $X_4 = \begin{pmatrix} & x_1 \\ x_2 & \end{pmatrix}$,
and for $i \le m-1$,

$$(-1)^{m+i} d_{ii} = (-1)^{m+i+1}\pi + (-1)^m \pi^{-m} c_S$$

$$(-1)^{m+i+1} a_{i^\vee i^\vee} = (-1)^{m+i}\pi + (-1)^m \pi^{-m} c_S$$

then $a_{i^\vee i^\vee} + d_{ii} = -2\pi$.

When $S = \{1, \ldots, \hat{j}, \ldots, n, n+i\}, j \neq i, i < j^\vee$,
for $i < j^\vee \le m-1$,

$$WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) = (-1)^{m+1}\pi^{-(m-1)}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j+1} e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})$$

for $i \le m-1, j^\vee \ge m+2$,

$$WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) = (-1)^{m}\pi^{-m}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j} e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})$$

for $m+2 \le i < j^\vee$,

$$WT(g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}) = (-1)^{m}\pi^{-(m+1)}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j+1} e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}})$$

These are the only basis elements involving such e_S's. As in §4, together with the previous paragraph they imply

$$B = B^{\text{ad}}, C = C^{\text{ad}}, D = -2\pi I_{m-1} - A^{\text{ad}} \tag{2.3.3}$$

for $i < m \le j^\vee \le m+1$,

$$g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp} = (-1)^{i+j+m}\pi^{-m}(e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}} + (-1)^{i+j+1}\pi e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}}) + \text{other terms}$$

for $m \le i \le m+1 < j^\vee$,

$$g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp} = (-1)^{m}\pi^{-(m+1)}(e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j}\pi e_{\{j^\vee,n+1,\ldots,\hat{i^*},\ldots,2n\}}) + \text{other terms}$$

Again those are the only basis elements involving such e_S's, it follows from the coefficients that

$$m_{i,1} = \pi e_{2,i^\vee-(m+1)}, m_{i,2} = \pi e_{1,i^\vee-(m+1)},$$

or

$$M = \pi E^{\mathrm{ad}} \tag{2.3.4}$$

$$l_{i^\vee-(m+1),1} = -\pi f_{2,i}, l_{i^\vee-(m+1),2} = -\pi f_{1,i},$$

or

$$L = -\pi F^{\mathrm{ad}} \tag{2.3.5}$$

(2.3.1) combined with (2.3.3),

$$J_{n-2}X_1 = \begin{pmatrix} HC & HD \\ -HA & -HB \end{pmatrix} = \begin{pmatrix} C^tH & -A^tH - 2\pi H \\ D^tH + 2\pi H & -B^tH \end{pmatrix} = -X_1^t J_{n-2} - 2\pi J_{n-2}.$$

so

$$X_1 = -\frac{1}{2} J_{n-2} X_3^t H_2 X_3 - \pi I_{n-2},$$

i.e.

$$A = -\frac{1}{2}F^{ad}E - \pi I_{m-1}, B = -\frac{1}{2}F^{ad}F, C = \frac{1}{2}E^{ad}E, D = \frac{1}{2}E^{ad}F - \pi I_{m-1}, \tag{2.3.6}$$

If $\pi = 0$, under (2.3.2), (2.3.4), (2.3.5), (2.3.6) together with the wedge condition, we've recovered (1.4.22)-(1.4.24), in other words the other closed conditions on $\mathcal{F}_i$ are trivial on this subfunctor. The existence of $\mathcal{F}_i^0$ garanteeing the wedge condition, all we need is to take $\mathcal{F}_i^0$ into account as well.

$$\mathcal{F}_{m-1} = \mathrm{colspan} \begin{pmatrix} A & B & -\pi F_2^{\mathrm{ad}} & -\pi F_1^{\mathrm{ad}} \\ C & D & \pi E_2^{\mathrm{ad}} & \pi E_1^{\mathrm{ad}} \\ E_1 & F_1 & & x_1 \\ E_2 & F_2 & x_2 & \\ I_{m-1} & & & \\ & I_{m-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\mathcal{F}_m = \operatorname{colspan}\begin{pmatrix} A & B & -F_1^{\mathrm{ad}} & -\pi F_1^{\mathrm{ad}} \\ C & D & E_1^{\mathrm{ad}} & \pi E_1^{\mathrm{ad}} \\ & & x_1 & \\ E_2 & F_2 & 1 & \\ I_{m-1} & & & \\ & I_{m-1} & & \\ E_1 & F_1 & & x_1 \\ & & & 1 \end{pmatrix}$$

$$\mathcal{F}_{m+1} = \operatorname{colspan}\begin{pmatrix} -A & -B & -F_2^{\mathrm{ad}} & -F_1^{\mathrm{ad}} \\ -C & -D & E_2^{\mathrm{ad}} & E_1^{\mathrm{ad}} \\ -\pi E_1 & -\pi F_1 & & x_1 \\ -\pi E_2 & -\pi F_2 & x_2 & \\ I_{m-1} & & & \\ & I_{m-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

with respect to corresponding bases respectively, then

$$\mathcal{F}_m^0 = \operatorname{colspan}\begin{pmatrix} -\pi F_1^{\mathrm{ad}} \\ \pi E_1^{\mathrm{ad}} \\ \pi x_1 \\ \pi \\ -F_1^{\mathrm{ad}} \\ E_1^{\mathrm{ad}} \\ x_1 \\ 1 \end{pmatrix}$$

$(\pi \otimes 1 + 1 \otimes \pi)\mathcal{F}_{m-1} \subset \mathcal{F}_{m-1}^0$ and $(\pi \otimes 1 + 1 \otimes \pi)\mathcal{F}_{m+1} \subset \mathcal{F}_{m+1}^0$ translate to

$$\operatorname{colspan}\begin{pmatrix} A + \pi I_{m-1} & B & -\pi F_2^{\mathrm{ad}} & -\pi F_1^{\mathrm{ad}} \\ C & D + \pi I_{m-1} & \pi E_2^{\mathrm{ad}} & \pi E_1^{\mathrm{ad}} \\ E_1 & F_1 & \pi & x_1 \\ E_2 & F_2 & x_2 & \pi \end{pmatrix} \subset \operatorname{colspan}\begin{pmatrix} s_1 \\ \vdots \\ s_{n-1} \\ s_n \end{pmatrix}, \text{ and}$$

$$\mathrm{colspan}\begin{pmatrix} -A+\pi I_{m-1} & -B & -F_2^{\mathrm{ad}} & -F_1^{\mathrm{ad}} \\ -C & -D+\pi I_{m-1} & E_2^{\mathrm{ad}} & E_1^{\mathrm{ad}} \\ -\pi E_1 & -\pi F_1 & \pi & x_1 \\ -\pi E_2 & -\pi F_2 & x_2 & \pi \end{pmatrix} \subset \mathrm{colspan}\begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ t_n \end{pmatrix},$$

$T_{m-1,m}(\mathcal{F}^0_{m-1}) \subset \mathcal{F}^0_m$ translates to $\begin{pmatrix} s_1 \\ \vdots \\ \pi s_{n-1} \\ s_n \end{pmatrix} = s_n \begin{pmatrix} -F_1^{\mathrm{ad}} \\ E_1^{\mathrm{ad}} \\ x_1 \\ 1 \end{pmatrix}$

$T_{m+1,n+m-1}(\mathcal{F}^0_{m+1}) \subset \mathcal{F}^0_{n+m-1}$ translates to $\begin{pmatrix} \pi t_1 \\ \vdots \\ \pi t_{n-2} \\ t_{n-1} \\ t_n \end{pmatrix} \in \mathrm{colspan}\begin{pmatrix} s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \\ s_n \end{pmatrix}$

$s_1, \ldots, s_{n-2}$ are multiples to s_n, so we may assume either $s_{n-1} = 1$ or $s_n = 1$.

Clearly the data (S, T) are equivalent to the row vector and column vector of the rank 1 matrix $\begin{pmatrix} E_1 & F_1 & \pi & x_1 \\ E_2 & F_2 & x_2 & \pi \end{pmatrix}$ compatible with the wedge condition, defining a blow-up along the point z_0.

If $s_{n-1} = 1$, $\begin{pmatrix} E_2 & F_2 & x_2 & \pi \end{pmatrix} = s_n \begin{pmatrix} E_1 & F_1 & \pi & x_1 \end{pmatrix}$, and $\begin{pmatrix} \pi t_1 \\ \vdots \\ \pi t_{n-2} \\ t_{n-1} \\ t_n \end{pmatrix} = t_{n-1} \begin{pmatrix} s_1 \\ \vdots \\ s_{n-2} \\ 1 \\ s_n \end{pmatrix}$

Since $t_n = t_{n-1}s_n$, either $t_{n-1} = 1$ or $t_i = 1$ for some $1 \leq i \leq n-2$.

If $t_{n-1} = 1, s_n = t_n$, $\begin{pmatrix} -F_1^{\mathrm{ad}} \\ E_1^{\mathrm{ad}} \end{pmatrix} = x_1 \begin{pmatrix} t_1 \\ \vdots \\ t_{n-2} \end{pmatrix}$,

all said, we can conclude

$$\tilde{M}_{\{m-1,m\}} \cap U_{n-1,n-1} = \mathrm{Spec}\ \mathcal{O}_F[x_1, t_1, \ldots, t_{n-2}, t_n]/(x_1 t_n - \pi). \tag{2.3.7}$$

And furthermore $\tilde{Z}_0 \cap U_{n-1,n-1}$ is the locus $x_1 = 0$,

$\tilde{Z}_2 \cap U_{n-1,n-1}$ is the locus $t_n = 0$, both Cartier divisors, while $\tilde{Z}_1 \cap U_{n-1,n-1} = \tilde{Z}_3 \cap U_{n-1,n-1} = \emptyset$.

$$\text{If } t_1 = 1, \begin{pmatrix} -F_1^{\mathrm{ad}} \\ E_1^{\mathrm{ad}} \\ x_1 \\ \pi \end{pmatrix} = (-f_{1,m-1}) \begin{pmatrix} 1 \\ \vdots \\ t_{n-1} \\ t_n \end{pmatrix},$$

$$\tilde{M}_{\{m-1,m\}} \cap U_{n-1,1} = \mathrm{Spec}\ \mathcal{O}_F[f_{1,m-1}, t_2, \ldots, t_{n-1}, s_n]/(f_{1,m-1}t_{n-1}s_n + \pi). \tag{2.3.8}$$

$\tilde{Z}_0 \cap U_{n-1,1}$ is the locus $f_{1,m-1} = 0$,

$\tilde{Z}_1 \cap U_{n-1,1}$ is the locus $t_{n-1} = 0$,

$\tilde{Z}_2 \cap U_{n-1,1}$ is the locus $s_n = 0$, all Cartier divisors,

while $\tilde{Z}_3 \cap U_{n-1,1} = \emptyset$.

In all the other affine neighborhoods presented equations of the same forms, thanks to symmetry. These local equations also verify our decomposition of the special fiber and illustrate that $\tilde{Z}_2$ and $\tilde{Z}_3$ are disjoint. It's clear from (2.3.7) and (2.3.8) that flatness and semi-stable reduction hold and regularity follows.

□

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Curriculum Vitae

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